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Research Article

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Degree bounds for modular covariants

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Abstract: Let V, W be representations of a cyclic group G of prime order p over a field $\Bbbk$ of characteristic p. The module of covariants $\Bbbk[V, W]^G$ is the set of G-equivariant polynomial maps $V \to W$, and is a module over $\Bbbk[V]^G$. We give a formula for the Noether bound $\beta(\Bbbk[V, W]^G, \Bbbk[V]^G)$, i.e. the minimal degree d such that $\Bbbk[V, W]^G$ is generated over $\Bbbk[V]^G$ by elements of degree at most d.

Keywords: Invariant theory, modular representation, cyclic group, module of covariants, Noether bound

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1 Introduction

Let G be a finite group, $\Bbbk$ a field and V, W a pair of finite-dimensional $\Bbbk G$-modules. Let $\Bbbk[V]$ denote the symmetric algebra on the dual V^* of V and let $\Bbbk[V, W] = \Bbbk[V] \otimes_{\Bbbk} W$. Elements of $\Bbbk[V]$ represent polynomial functions $V \to \Bbbk$ and elements of $\Bbbk[V, W]$ represent polynomial functions $V \to W$; for $f \otimes w \in \Bbbk[V, W]$ the corresponding function takes v to $f(v)w$. The group G acts by algebra automorphisms on $\Bbbk[V]$ and hence diagonally on $\Bbbk[V, W]$. The fixed points $\Bbbk[V, W]^G$ of this action are called covariants and represent G-equivariant polynomial functions $V \to W$. The the fixed points $\Bbbk[V]^G$ are called invariants. For $f \in \Bbbk[V]^G$ and $\phi \in \Bbbk[V, W]^G$ we define the product

$$f\phi(v) = f(v)\phi(v).$$

Then $\Bbbk[V]^G$ is a $\Bbbk$-algebra and $\Bbbk[V, W]^G$ is a finite $\Bbbk[V]^G$-module. Modules of covariants in the non-modular case ($|G| \neq 0 \in \Bbbk$) were studied by Chevalley [3], Shephard–Todd [10], Eagon–Hochster [7]. In the modular case far less is known, but recent work of Broer and Chuai [1] has shed some light on the subject. A systematic attempt to construct generating sets for modules of covariants when G is a cyclic group of order p was begun by the first author in [5].

Let $A = \bigoplus_{d \geq 0} A_d$ be any graded $\Bbbk$-algebra and $M = \sum_{d \geq 0} M_d$ any graded A-module, where A_d and M_d denote the d-th homogeneous components of A and M, respectively. Then the Noether bound $\beta(A)$ is defined to be the minimum degree $d > 0$ such that A is generated by the set $\{a : a \in A_k,\ k \leq d\}$. Similarly, $\beta(M, A)$ is defined to be the minimum degree $d > 0$ such that M is generated over A by the set $\{m : m \in M_k,\ k \leq d\}$, and we write $\beta(M) = \beta(M, A)$ when the context is clear.

Noether famously showed that $\beta(\mathbb{C}[V]^G) \leq |G|$ for arbitrary finite G, but computing Noether bounds in the modular case is highly nontrivial. When G is cyclic of prime order, the second author along with Fleischmann, Shank and Woodcock [6] determined the Noether bound for any $\Bbbk G$-module. The purpose of this article is to find results similar to those in [6] for covariants. Our main result can be stated concisely as follows.

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Theorem 1. *Let G be a cyclic group of order p, $\Bbbk$ a field of characteristic p, V a reduced $\Bbbk G$-module and W a nontrivial indecomposable $\Bbbk G$-module. Then*

$$\beta(\Bbbk[V,W]^G) = \beta(\Bbbk[V]^G)$$

unless V is indecomposable of dimension 2.

Here by *reduced* we mean that the direct sum decomposition of V contains no summands on which G acts trivially; see also remarks following Proposition 4.

2 Preliminaries

For the rest of this article, G denotes a cyclic group of order $p > 0$, and we let $\Bbbk$ be a field of characteristic p. We choose a generator σ for G. Over $\Bbbk$, there are p indecomposable representations $V_1, \ldots, V_p$ and each indecomposable representation V_i is afforded by a Jordan block of size i. Note that V_p is isomorphic to the free module $\Bbbk G$, and this is the unique free indecomposable $\Bbbk G$ -module.

Let $\Delta = \sigma - 1 \in \Bbbk G$. We define the transfer map $\mathrm{Tr} : \Bbbk[V] \to \Bbbk[V]$ by $\sum_{1\le i\le p} \sigma^i$. Notice that we also have $\mathrm{Tr} = \Delta^{p-1}$. Invariants that are in the image of Tr are called transfers.

Remark 2. Let $e_1, \ldots, e_i$ be an upper triangular basis for the i-dimensional indecomposable representation V_i. Then $\Delta(e_j) = e_{j-1}$ for $2 \le j \le i$ and $\Delta(e_1) = 0$. Therefore $\Delta^j(V_i) = 0$ for all $j \ge i$. Note that for an indecomposable module V_i we have $\Delta(V_i) \cong V_{i-1}$ for $2 \le i \le p$ and $\Delta(V_1) = 0$. It follows that an invariant f is in the image of the linear map $\Delta^j : \Bbbk[V] \to \Bbbk[V]$ if and only if it is a linear combination of fixed points in indecomposable modules of dimension at least $j + 1$. In particular, an invariant is in the image of the transfer map $(= \Delta^{p-1})$ if and only if it is a linear combination of fixed points of free $\Bbbk G$-modules.

We assume that V and W are $\Bbbk G$-modules with W indecomposable and we choose a basis $w_1, \ldots, w_n$ for W so that we have

$$\sigma w_i = \sum_{1\le j\le i} (-1)^{i-j} w_j$$

for $1 \le i \le n$. For $f \in \Bbbk[V]$ we define the *weight* of f to be the smallest positive integer d with $\Delta^d(f) = 0$. Note that $\Delta^p = (\sigma - 1)^p = 0$, so the weight of a polynomial is at most p.

A useful description of covariants is given in [5]. We include this description here for completeness.

Proposition 3 ([5, Proposition 3]). *Let $f \in \Bbbk[V]$ with weight $d \le n$. Then*

$$\sum_{1\le j\le d} \Delta^{j-1}(f) w_j \in \Bbbk[V,W]^G.$$

Conversely, if

$$f_1 w_1 + f_2 w_2 + \cdots + f_n w_n \in \Bbbk[V,W]^G,$$

then there exists $f \in \Bbbk[V]$ with weight $\le n$ such that $f_j = \Delta^{j-1}(f)$ for $1 \le j \le n$.

For a non-zero covariant $h = f_1 w_1 + f_2 w_2 + \cdots + f_n w_n$, we define the *support* of h to be the largest integer j such that $f_j \neq 0$. We denote the support of h by $s(h)$. We shall say h is a *transfer covariant* if there exists a non-negative integer k and $f \in \Bbbk[V]$ such that $f_1 = \Delta^k(f), f_2 = \Delta^{k+1}(f), \ldots, f_{s(h)} = \Delta^{p-1}(f)$ for some $f \in \Bbbk[V]$.

We call a homogeneous invariant in $\Bbbk[V]^G$ indecomposable if it is not in the subalgebra of $\Bbbk[V]^G$ generated by invariants of strictly smaller degree. Similarly, a homogeneous covariant in $\Bbbk[V,W]^G$ is indecomposable if it does not lie in the submodule of $\Bbbk[V,W]^G$ generated by covariants of strictly smaller degree.

3 Upper bounds

We first prove a result on decomposability of a transfer covariant. In the proof below we set $\gamma = \beta(\Bbbk[V], \Bbbk[V]^G)$.

Proposition 4. *Let $f \in \Bbbk[V]$ be homogeneous and let $h = \Delta^k(f)w_1 + \Delta^{k+1}(f)w_2 + \cdots + \Delta^{p-1}(f)w_{s(h)}$ be a transfer covariant of degree $> \gamma$. Then h is decomposable.*

Proof. Let $g_1, \ldots, g_t$ be a set of homogeneous polynomials of degree at most γ generating $\Bbbk[V]$ as a module over $\Bbbk[V]^G$. So we can write $f = \sum_{1\le i\le t} q_i g_i$, where each $q_i \in \Bbbk[V]^G_+$ is a positive degree invariant. Since Δ^j is $\Bbbk[V]^G$-linear, we have $\Delta^j(f) = \sum_{1\le i\le t} q_i \Delta^j(g_i)$ for $k \le j \le p-1$. It follows that

$$h = \sum_{1\le i\le t} q_i(\Delta^k(g_i)w_1 + \cdots + \Delta^{p-1}(g_i)w_{s(h)}).$$

Note that $\Delta^k(g_i)w_1 + \cdots + \Delta^{p-1}(g_i)w_{s(h)}$ is a covariant for each $1 \le i \le t$ by Proposition 3. We also have $q_i \in \Bbbk[V]^G_+$ so it follows that h is decomposable. □

Write $V = \bigoplus_{j=1}^m V_{n_j}$ as a sum of indecomposable modules. Note that

$$\Bbbk[V \oplus V_1, W]^G = (S(V^*) \otimes S(V_1^*)) \otimes W)^G = \Bbbk[V, W]^G \otimes \Bbbk[V_1].$$

Therefore we will assume that $n_j > 1$ for all j; such representations are called reduced. Choose a basis $\{x_{i,j} : 1 \le i \le n_j,\ 1 \le j \le m\}$ for V^*, with respect to which we have

$$\sigma(x_{i,j}) = \begin{cases} x_{i,j} + x_{i+1,j}, & i < n_j, \\ x_{i,j}, & i = n_j. \end{cases}$$

This induces a multidegree on $\Bbbk[V] = \bigoplus_{\mathbf{d}\in\mathbb{N}^m} \Bbbk[V]_{\mathbf{d}}$ which is compatible with the action of G. For $1 \le j \le m$ we define $N_j = \prod_{k=0}^{p-1} \sigma^k x_{1,j}$, and note that the coefficient of $x_{1,j}^p$ in N_j is 1. Given any $f \in \Bbbk[V_{n_j}]$, we can therefore perform long division, writing

$$f = q_j N_j + r,$$

where $q_j \in \Bbbk[V_{n_j}]$ for all j and $r \in \Bbbk[V_{n_j}]$ has degree $< p$ in the variable $x_{1,j}$. This induces a vector space decomposition

$$\Bbbk[V_{n_j}] = N_j \Bbbk[V_{n_j}] \oplus B_j,$$

where B_j is the subspace of $\Bbbk[V_{n_j}]$ spanned by monomials with $x_{1,j}$-degree $< p$, but the form of the action implies that B_j and its complement are $\Bbbk G$-modules, so we obtain a $\Bbbk G$-module decomposition. Since $\Bbbk[V] = \bigotimes_{j=1}^m \Bbbk[V_{n_j}]$, it follows that

$$\Bbbk[V] = N_j \Bbbk[V] \oplus (B_j \otimes \Bbbk[V']),$$

where $V' = V_{n_1} \oplus \cdots \oplus V_{n_{j-1}} \oplus V_{n_{j+1}} \cdots \oplus V_{n_m}$. From this decomposition it follows that if M is a $\Bbbk G$ direct summand of $\Bbbk[V]_d$, then $N_j M$ is a $\Bbbk G$ direct summand of $\Bbbk[V]_{d+p}$ with the same isomorphism type. Further, any $f \in \Bbbk[V]^G$ can be written as

$$f = qN_j + r$$

with $q \in \Bbbk[V]^G$ and $r \in (B_j \otimes \Bbbk[V'])^G$. If in addition $\deg(f) = (d_1, d_2, \ldots, d_m)$ with $d_j > p - n_j$, then the degree d_j homogeneous component of B_j is free by [8, 2.10] and since tensoring a module with a free (projective) module gives a free (projective) module we may further assume, by Remark 2, that r is in the image of the transfer map.

If $h = \sum_{i=1}^{s(h)} \Delta^{i-1}(f)w_i \in \Bbbk[V, W]^G$, we define the multidegree of h to be that of f. Since G preserves the multidegree, this is the same as the multidegree of $\Delta^{i-1}(f)$ for all $i \le s(h)$. Then the analogue of this result for covariants is the following:

Proposition 5. *Let h be a covariant of multidegree $d_1, d_2, \ldots, d_m$ with $d_j > p - n_j$ for some j. Then there exist a covariant h_1 and a transfer covariant h_2 such that $h = N_j h_1 + h_2$.*

Proof. We proceed by induction on the support $s(h)$ of h. If $s(h) = 1$, then by Proposition 3, we have that $h = fw_1$ with $f \in \Bbbk[V]^G$. Then we can write $f = qN_j + \Delta^{p-1}(t)$ for some $q \in \Bbbk[V]^G$ and $t \in \Bbbk[V]$. Then both qw_1 and $\Delta^{p-1}(t)w_1$ are covariants by Proposition 3 and therefore $h = qN_j w_1 + \Delta^{p-1}(t)w_1$ gives us the desired decomposition.

Now assume that $s(h) = k$. Then by Proposition 3 there exists $f \in \Bbbk[V]$ such that

$$h = fw_1 + \Delta(f)w_2 + \cdots + \Delta^{k-1}(f)w_k,$$

with $\Delta^k(f) = 0$. Since $\Delta^{k-1}(f) \in \Bbbk[V]^G$ and $d_j > p - n_j$, we can write $\Delta^{k-1}(f) = qN_j + \Delta^{p-1}(t)$ for some $q \in \Bbbk[V]^G$ and $t \in \Bbbk[V]$. It follows that qN_j is in the image of Δ^{k-1}. But since multiplication by N_j preserves the isomorphism type of a module, it follows that q is in the image of Δ^{k-1}. Write $q = \Delta^{k-1}(f')$ with $f' \in \Bbbk[V]$. Set

$$h_1 = f'w_1 + \Delta(f')w_2 + \cdots + \Delta^{k-1}(f')w_k \quad \text{and} \quad h_2 = \Delta^{p-k}(t)w_1 + \cdots + \Delta^{p-1}(t)w_k.$$

Since $\Delta^{k-1}(f') \in \Bbbk[V]^G$, it follows that h_1 is a covariant by Proposition 3. Consider the covariant

$$h' = h - N_j h_1 - h_2.$$

Since $\Delta^{k-1}(f) = \Delta^{p-1}(t) + \Delta^{k-1}(f')N_j$, the support of h' is strictly smaller than the support of h. Moreover, h_2 is a transfer covariant and so the assertion of the proposition follows by induction. □

We obtain the following upper bound for the Noether number of covariants:

Proposition 6. *We have* $\beta(\Bbbk[V,W]^G) \leq \max(\beta(\Bbbk[V], \Bbbk[V]^G), mp - \dim(V))$.

Proof. Let $h \in \Bbbk[V,W]^G$ with degree $d > \max(\beta(\Bbbk[V], \Bbbk[V]^G), mp - \dim(V))$. Let $(d_1, d_2, \ldots, d_m)$ be the multidegree of h. Then we must have $d_j > p - n_j$ for some j. Consequently, we may apply Proposition 5, writing

$$h = N_j h_1 + h_2,$$

where h_2 is a transfer covariant. Since $\deg(h_2) > \beta(\Bbbk[V], \Bbbk[V]^G)$, it follows that h_2 is decomposable by Proposition 4, and so we have shown that h is decomposable. □

4 Lower bounds

Indecomposable transfers are one method of obtaining lower bounds for $\beta(\Bbbk[V]^G)$. Recall that we have written $V = \bigoplus_{j=1}^m V_{n_j}$ as a sum of indecomposable modules. The analogous result for covariants is:

Lemma 7. *Let $n \geq 2$ and let $\Delta^{p-1}(f) \in \Bbbk[V]^G$ be an indecomposable homogeneous transfer. Then the transfer covariant*

$$h = \Delta^{p-n}(f)w_1 + \cdots + \Delta^{p-1}(f)w_n$$

is indecomposable.

Proof. Assume on the contrary that h is decomposable. Then there exist homogeneous $q_i \in \Bbbk[V]^G_+$ and $h_i \in \Bbbk[V,W]^G$ such that $h = \sum_{1\leq i\leq t} q_i h_i$. Write $h_i = h_{i,1}w_1 + \cdots + h_{i,n}w_n$ for $1 \leq i \leq t$. Then we have

$$\Delta^{p-1}(f) = \sum_{1\leq i\leq t} q_i h_{i,n}.$$

By Proposition 3 we have $\Delta(h_{i,n-1}) = h_{i,n}$ and so $h_{i,n} \in \Bbbk[V]^G_+$ because $n \geq 2$. It follows that $\sum_{1\leq i\leq t} q_i h_{i,n}$ is a decomposition of $\Delta^{p-1}(f)$ in terms of invariants of strictly smaller degree, contradicting the indecomposability of $\Delta^{p-1}(f)$. □

Corollary 8. *Suppose $n \geq 2$ and $\beta(\Bbbk[V]^G) > \max(p, mp - \dim(V))$. Then $\beta(\Bbbk[V]^G) \leq \beta(\Bbbk[V,W]^G)$.*

Proof. By [8, Lemma 2.12], $\Bbbk[V]^G$ is generated by the norms $N_1, N_2, \ldots, N_m$, invariants of degree at most $mp - \dim(V)$, and transfers. Since there exists an indecomposable invariant of degree $\beta(\Bbbk[V]^G)$, if the hypotheses of the corollary above hold, then $\Bbbk[V]^G$ contains an indecomposable transfer with this degree. By Lemma 7, $\Bbbk[V,W]^G$ contains a transfer covariant of degree $\beta(\Bbbk[V]^G)$ which is indecomposable, from which the conclusion follows. □

5 Main results

We are now ready to prove Theorem 1. Note that $\Bbbk[V, V_1]^G$ is generated over $\Bbbk[V]^G$ by w_1 alone, which has degree zero, and therefore $\beta(\Bbbk[V, V_1]^G) = 0$. For this reason we assume $n \geq 2$ throughout.

Proof. Suppose first that $n_j > 3$ for some j. Then by [6, Proposition 1.1 (a)], we have

$$\beta(\Bbbk[V]^G) = m(p-1) + (p-2).$$

Since V is reduced, we have $\dim(V) \geq 2m$ and hence

$$\beta(\Bbbk[V]^G) > m(p-2) \geq mp - \dim(V).$$

Also, $\beta(\Bbbk[V]^G) \geq 2p - 3 > p$ since $n_j \leq p$ for all j. Therefore Corollary 8 implies that $\beta(\Bbbk[V]^G) \leq \beta(\Bbbk[V, W]^G)$. On the other hand, [6, Lemma 3.3] shows that the top degree of $\Bbbk[V]/\Bbbk[V]^G_+\Bbbk[V]$ is bounded above by $m(p-1) + (p-2)$. By the graded Nakayama Lemma it follows that $\beta(\Bbbk[V], \Bbbk[V]^G) \leq m(p-1) + (p-2)$. We have already shown that this number is at least $mp - \dim(V) + 1$, so by Proposition 6 we get that

$$\beta(\Bbbk[V, W]^G) \leq m(p-1) + (p-2) = \beta(\Bbbk[V]^G)$$

as required.

Now suppose that $n_i \leq 3$ for all i and $n_j = 3$ for some j. Then by [6, Proposition 1.1 (b)], we have

$$\beta(\Bbbk[V]^G) = m(p-1) + 1.$$

Since V is reduced, we have $\dim(V) \geq 2m$ and hence

$$\beta(\Bbbk[V]^G) > m(p-2) \geq mp - \dim(V).$$

Also $\beta(\Bbbk[V]^G) \geq 2p - 1 > p$ provided $m \geq 2$. In that case Corollary 8 applies. If $m = 1$, then Dickson [4] has shown that $\Bbbk[V]^G = \Bbbk[x_1, x_2, x_3]^G$ is minimally generated by the invariants x_3, $x_2^2 - 2x_1x_3 - x_2x_3$, N, $\Delta^{p-1}(x_1^{p-1}x_2)$. It follows that $\Delta^{p-1}(x_1^{p-1}x_2)$ is an indecomposable transfer, so by Lemma 7, $\Bbbk[V, W]^G$ contains an indecomposable transfer covariant of degree $p = \beta(\Bbbk[V]^G)$. In either case we obtain

$$\beta(\Bbbk[V, W]^G) \geq \beta(\Bbbk[V]^G).$$

On the other hand, by [9, Corollary 2.8], $m(p-1) + 1$ is an upper bound for the top degree of $\Bbbk[V]/\Bbbk[V]^G_+$. By the same argument as before we get $\beta(\Bbbk[V]^G, \Bbbk[V]) \leq m(p-1) + 1$. We have already shown that this number is at least $mp - \dim(V) + 1$, so by Proposition 6 we get that

$$\beta(\Bbbk[V, W]^G) \leq m(p-1) + 1 = \beta(\Bbbk[V]^G)$$

as required.

It remains to deal with the case $n_i = 2$ for all i, i.e. $V = mV_2$. We assume $m \geq 2$. In this case Campbell and Hughes [2] showed that $\beta(\Bbbk[V]^G) = (p-1)m$. As $\dim(V) = 2m$, we have $\beta(\Bbbk[V]^G) > m(p-2) = mp - \dim(V)$. If $m \geq 3$ or $m = 2$ and $p > 2$, then we have

$$\beta(\Bbbk[V]^G) > p$$

and Corollary 8 applies. In case $m = 2 = p$, $\Bbbk[V]^G = \Bbbk[x_{1,1}, x_{2,1}, x_{1,2}, x_{2,2}]^G$ is a hypersurface, minimally generated by $\{x_{2,1}, N_1, x_{2,2}, N_2, \Delta^{p-1}(x_{1,1}x_{1,2})\}$. In particular, $\Delta^{p-1}(x_{1,1}x_{1,2})$ is an indecomposable transfer, so by Lemma 7, $\Bbbk[V, W]^G$ contains an indecomposable transfer covariant of degree 2. In both cases we get

$$\beta(\Bbbk[V, W]^G) \geq \beta(\Bbbk[V]^G).$$

On the other hand, by [9, Theorem 2.1], the top degree of $\Bbbk[V]/\Bbbk[V]^G_+\Bbbk[V]$ is bounded above by $m(p-1)$. We have already shown this number is at least $mp - \dim(V) + 1$. Therefore, by Proposition 6, we get

$$\beta(\Bbbk[V, W]^G) \leq \beta(\Bbbk[V]^G)$$

as required. □

Remark 9. The only reduced representation not covered by Theorem 1 is $V = V_2$. An explicit minimal set of generators of $\Bbbk[V_2, W]^G$ as a module over $\Bbbk[V_2]^G$ is given in [5], the result is

$$\beta(\Bbbk[V_2, W]) = n - 1.$$

This is the only situation in which the Noether number is seen to depend on W.

Remark 10. Suppose V is any reduced $\Bbbk G$-module and $W = \bigoplus_{i=1}^r W_i$ is a decomposable $\Bbbk G$-module. Then

$$\Bbbk[V, W]^G = (S(V^*) \otimes \left(\bigoplus_{i=1}^{r} W_i)\right)^G = \bigoplus_{i=1}^{r}(S(V^*) \otimes W_i)^G.$$

So $\beta(\Bbbk[V, W]^G) = \max\{\beta(\Bbbk[V, W_i]^G) : i = 1, \dots, r\} = \beta(\Bbbk[V]^G)$ unless V is indecomposable of dimension 2, in which case we have

$$\beta(\Bbbk[V_2, W]^G) = \max\{\beta(\Bbbk[V_2, W_i]^G) : i = 1, \dots, r\} = \max\{\dim(W_i) - 1 : i = 1, \dots, r\}.$$

Thus, the results of this paper can be used to compute $\beta(\Bbbk[V, W]^G)$ for arbitrary $\Bbbk G$-modules V and W.

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